Stochastic Network Interdiction

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Purpose of this talk

- Describe a new, simple solution method for two-stage stochastic integer programs
- Motivate and illustrate with a particular stochastic, network-interdiction problem (will consider one other, too)
- Illustrate two main thrusts of my research, interdiction and SP.

First: Other work

- Interdiction of communications networks: Physical and cyber-attacks
- General models and solutions for system interdiction and defense
- General theoretical work on SPs
- Applications of SP: Sealift deployments subject to bio-attacks
- Integer programming

Generic network interdiction problem

- Using limited resources, attack an adversary's network so as to minimize the functionality of that network (to the adversary).
- Networks: Road, pipeline, comm
- Functionality: <u>Max</u> <u>flow</u>, shortest path, convoy movement, path existence
- Attacks: Aerial sorties, cruise missiles, special operations
- Can generalize: "system interdiction"

Max-flow interdiction

Basic Deterministic Model

on *G*=(*N*,*A*) with artificial arc *a* = (*t*,*s*)

$$z^* = \min_{\mathbf{x}\in X} \max_{\mathbf{y}} y_a$$

$$\text{s.t.} \sum_{k\in FS(i)} y_k - \sum_{k\in RS(i)} y_k = 0 \quad \forall i \in N$$

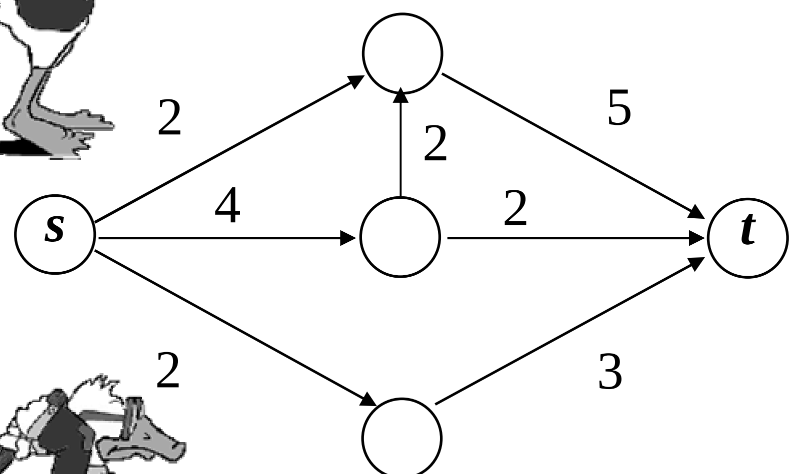
$$0 \le y_k \le u_k(1-x_k) \ \forall k \in A-a$$

where $X = \left\{\mathbf{x}\in\{0,1\}^{|A|} \mid R\mathbf{x} \le \mathbf{r}\right\}$ **, and…**

A Simple Example

Suppose we have enough resource to interduck any two arcs

***u_k* on arcs**



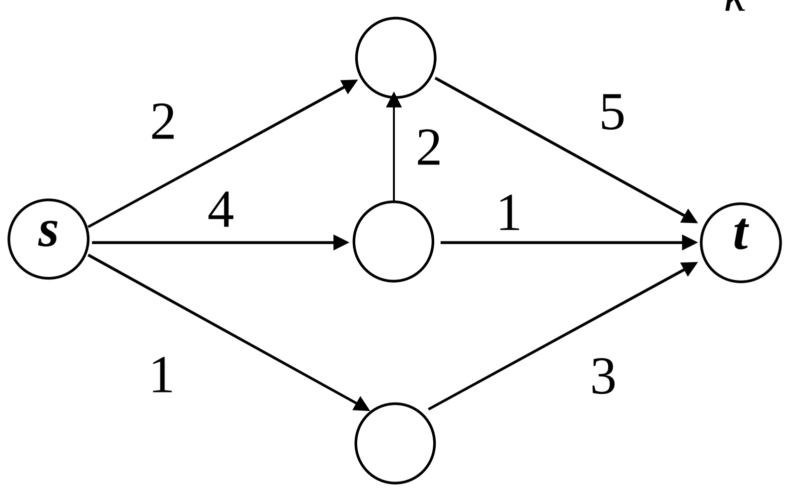
Max-flow interdiction

Converts to a MIP (well, IP actually)

$$\min_{\mathbf{x}\in X} \min_{\boldsymbol{\pi},\boldsymbol{\theta}} \sum_{k\in A-a} u_k \theta_k$$

$$\text{s.t.} \quad \pi_i - \pi_j + x_k + \theta_k \geq \begin{cases} 1 & if \;\; k = a \\ 0 & \forall\, k = (i,j) \in A - a \end{cases}$$

$$\theta_k \geq 0 \;\; \forall\, k \in A - a$$



Interdiction under uncertainty

- **Uncertain <u>success</u> or data, SMFI:**

$$\min_{\mathbf{x}\in X} Eh\left(\mathbf{x},\tilde{\mathbf{I}}\right)$$

$$\text{where } h\left(\mathbf{x},\tilde{\mathbf{I}}\right) \equiv \max_{\mathbf{y}\geq\mathbf{0}} y_a$$

$$\text{s.t.} \sum_{k\in FS(i)} y_k - \sum_{k\in RS(i)} y_k = 0 \quad \forall i \in N$$

$$0 \leq y_k \leq u_k(1-\tilde{I}_k x_k) \forall k \in A-a$$

$$\text{where } \tilde{I}_k = \begin{cases} 1 \text{ if interdiction of } k \text{ is successful} \\ 0 \text{ otherwise} \end{cases}$$

Alternative formulation

$$\min_{\mathbf{x}\in X} Eg\left(\mathbf{x},\tilde{\mathbf{I}}\right)$$

$$\text{where}\quad g\left(\mathbf{x},\tilde{\mathbf{I}}\right) \equiv \max_{\mathbf{y}\geq\mathbf{0}} y_a - \sum_{k\in A-a} x_k \tilde{I}_k y_k$$

$$\text{s.t.} \sum_{k\in FS(i)} y_k - \sum_{k\in RS(i)} y_k = 0 \quad \forall i \in N$$

$$0 \leq y_k \leq u_k \forall k \in A-a$$

- **Note: Deterministic problems are NP-complete. It's #P-complete to evaluate E*h* or E*g* for fixed x: These stochastic problems are really hard.**

SMFI: An instance of a 2SSP

$$\min_{\mathbf{x}} Eg^{D}\left(\mathbf{x},\tilde{\boldsymbol{\xi}}\right)$$

$$\text{s.t.} \qquad A\mathbf{x} = \mathbf{b}$$

$$\mathbf{0} \le \mathbf{x} \le \mathbf{u} \ \text{ and integer}$$

where $g^{D}\left(\mathbf{x},\tilde{\boldsymbol{\xi}}\right) \equiv \tilde{\mathbf{c}}\mathbf{x} + \min_{\mathbf{y}\ge\mathbf{0}} \tilde{\mathbf{f}}\mathbf{y}$

$$\text{s.t.} \ \ \tilde{D}\mathbf{y} = \tilde{\mathbf{d}} + \tilde{B}\mathbf{x}$$

$$\mathbf{y} \ge \mathbf{0} \ \text{ (some may be integer)}$$

and where $\tilde{\boldsymbol{\xi}} \equiv \mathrm{vec}(\tilde{\mathbf{f}},\tilde{D},\tilde{\mathbf{d}},\tilde{B})$

$g^{D}\left(\mathbf{x},\tilde{\boldsymbol{\xi}}\right)$ **represents the dual of our** $g\left(\mathbf{x},\tilde{\mathbf{I}}\right)$

Probability of kill

- Assume $p_k = E[\tilde{I}_k]$ is known
- Weaponeers know this stuff!
- Well…

Bound on *z**, pessimistic

- **New soln methodology needs bounds**
- **From Jensen's inequality, obtain a global upper bound given a "good" $\hat{\mathbf{x}}$:**

$$z^* \le Eh\left(\hat{\mathbf{x}}, \tilde{\mathbf{I}}\right) \text{ for any } \hat{\mathbf{x}} \in X$$

$$\le h\left(\hat{\mathbf{x}}, E[\tilde{\mathbf{I}}]\right) \qquad\qquad (\equiv z'')$$

$$= \max y_a$$

$$\text{s.t. Flow balance in } \mathbf{y}$$

$$0 \le y_k \le u_k(1 - E[\tilde{I}_k]\hat{x}_k)\ \forall k \in A - a$$

- **Can also use probabilistic bounds**

Bound on *z**, pessimistic

- **Actually, because interdictions are binary, and successes/failures are binary in SMFI, we can reformulate the upper-bound problem and minimize that bound via a MIP.**

$$z^* \le \min_{\mathbf{x}\in X,\boldsymbol{\pi},\boldsymbol{\theta}} \sum_{k\in A-a} \left(u_k\theta_k + u_k(1-E[\tilde{I}_k])x_k \right)$$

$$\text{s.t.}\quad \pi_i - \pi_j + x_k + \theta_k \ge \begin{cases} 1 & \text{if } k=(t,s) \\ 0 & \forall k=(i,j)\in A-a \end{cases}$$

$$\theta_k \ge 0 \;\; \forall\, k\in A-a$$

Bound on *z**, optimistic

- **A lower bounding MIP:**

$$
\begin{aligned}
z^* &\equiv \min_{\mathbf{x}\in X} Eg\left(\mathbf{x},\tilde{\mathbf{I}}\right) \\
&\geq \min_{\mathbf{x}\in X} g\left(\mathbf{x}, E[\tilde{\mathbf{I}}]\right) \text{ because } g \text{ is convex in } \tilde{\mathbf{I}} \\
&= \min_{\mathbf{x}\in X,\boldsymbol{\pi},\boldsymbol{\theta}} \sum_{k\in A-a} u_k\theta_k \\
&\qquad \text{s.t.}\quad \pi_i - \pi_j + E[\tilde{I}_k]x_k + \theta_k \geq \begin{cases} 1 & \text{if } k=(t,s) \\ 0 & \forall k=(i,j)\in A-a \end{cases} \\
&\qquad\qquad \theta_k \geq 0 \ \ \forall k \in A-a
\end{aligned}
$$

Bounds on z^*: Comments

- **Bounds can be improved by expanding in terms of conditional probabilities, e.g., by conditioning on the number of successful interdictions.**
- **Can use probabilistic bounds; may be necessary for other 2SSPs.**
- **But, keep it simple for now.**

Solution methodology, outline

- ***BOUND:*** **Find a global upper bnd** $z'' \geq z^*$
- ***ENUMERATE*** **all solns** $\hat{\mathbf{x}}$ **s.t.** $z'(\hat{\mathbf{x}}) \leq z^* \leq z''$; **call these candidates** $\hat{\mathbf{x}} \in \mathcal{X}$
- ***SCREEN*** **the candidates (Monte Carlo and statistics) to identify the best, or the best few** $\hat{\mathbf{x}} \in \mathcal{X}^* \subseteq \mathcal{X}$
- ***TEST*** $\hat{\mathbf{x}} \in \mathcal{X}^*$ **to determine quality**
- **(Or maybe *Partially Enumerate, Then Screen: PETS. Or,* maybe *Bound, Enumerate, Then Screen: BETS.*)**

Fundamental theorem for PE

- Theorem 1: $\hat{\mathbf{x}}$ can be optimal for SMFI only if $g\left(\hat{\mathbf{x}}, E[\tilde{\mathbf{I}}]\right) \leq z''$.

  Proof: Obvious. *QED*
- Theorem leads to finding a set of candidate solutions $\hat{\mathbf{x}} \in \mathcal{X}$ using the algorithm on the next slide.
- For simplicity, assume that the set of feasible interdiction plans defined by *X* has cardinality constraint:

$$\sum_{k \in A} x_k = R$$

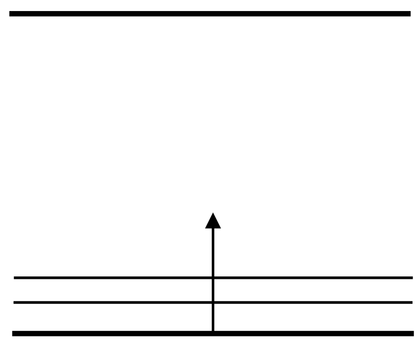
Alg. to find candidate solutions

1. $\mathcal{X} \leftarrow \emptyset$; Find a good $\hat{\mathbf{x}} \in X$;
2. Compute ub z'' given $\hat{\mathbf{x}}$; (or optimize)
3. Solve $z' = \min_{\mathbf{x} \in X} g\left(\mathbf{x}, E[\tilde{\mathbf{I}}]\right)$ for $\hat{\mathbf{x}}$;
4. If $(z' > z'')$ print $\mathcal{X}$ and halt;
5. Add $\hat{\mathbf{x}}$ to $\mathcal{X}$;
6. Add constraint $\sum_{k|\hat{x}_k=1} x_k \leq R-1$

   to constraint set X and go to 2;

Alg. to find candidate solutions

Find a good $\hat{\mathbf{x}} \in X$



z'' Compute $z'' = h\left(\hat{\mathbf{x}}, E[\tilde{\mathbf{I}}]\right)$

(Or optimize)

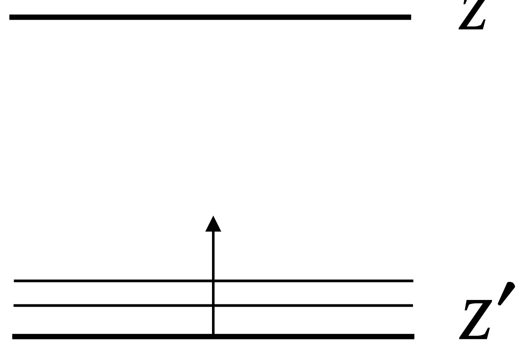
z' Compute $z' = \min\limits_{\substack{\mathbf{x} \in X \\ \mathbf{x} \neq \hat{\mathbf{x}}_1 \\ \mathbf{x} \neq \hat{\mathbf{x}}_2 \\ \vdots}} g\left(\mathbf{x}, E[\tilde{\mathbf{I}}]\right)$

For other 2SSPs, just use other bounds!

A better enumeration algorithm

Find a good $\hat{\mathbf{x}}$

Compute $z'' = h\left(\hat{\mathbf{x}}, E[\tilde{\mathbf{I}}]\right)$
(Or optimize)



Use B&B-like procedure to enumerate all

$$\hat{\mathbf{x}} \in X \text{ s.t. } z' = g\left(\hat{\mathbf{x}}, E[\tilde{\mathbf{I}}]\right) \le z''$$

Screening candidate solutions

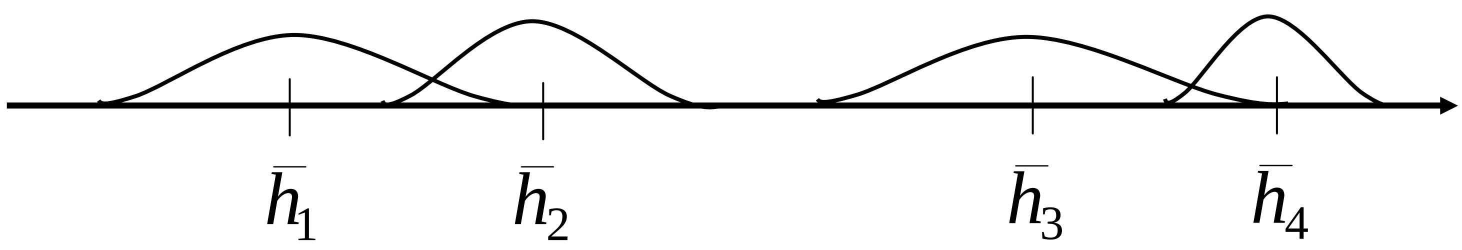
- **For small *R* we can compute $Eg\left(\hat{\mathbf{x}}_k, \tilde{\mathbf{I}}\right)$ exactly for each $\hat{\mathbf{x}}_k \in \mathcal{X}$: There are only 2^R ways for *R* attempted interdictions to succeed or fail. Can solve SMFI exactly in this case.**
- **Will describe general statistical screening procedures because they are necessary for most applications of BEST, including more complicated interdiction problems (and larger *R*).**
- **Seeking a near-optimal set $\mathcal{X}^* \subseteq \mathcal{X}$**

We could do this:

- **Sample the** $h\left(\hat{\mathbf{x}}_k, \tilde{\mathbf{I}}\right)$ **for each** $\hat{\mathbf{x}}_k \in \mathcal{X}$ **to obtain independent estimates**

$$\bar{h}_k = \frac{1}{L}\sum_{l=1}^{L} h(\hat{\mathbf{x}}_k, \hat{\mathbf{I}}_{kl}) \approx Eh\left(\hat{\mathbf{x}}_k, \tilde{\mathbf{I}}\right)$$

- **These** $\bar{h}_k$ **are distributed with independent t-distributions**
- **Reject** $\hat{\mathbf{x}}_k$ **that correspond to** $\bar{h}_k$ **being "too large"** (Reject $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_4$ below.)



But we will do this:

- **Using CRNs, sample the** $h\left(\hat{\mathbf{x}}_k, \tilde{\mathbf{I}}\right)$ **for each** $\hat{\mathbf{x}}_k \in \mathcal{X}$ **to obtain estimates**

$$\bar{h}_k = \frac{1}{L}\sum_{l=1}^{L} h(\hat{\mathbf{x}}_k, \hat{\mathbf{I}}_l) \approx Eh\left(\hat{\mathbf{x}}_k, \tilde{\mathbf{I}}\right)$$

- **Order:** $\bar{h}_1 \le \bar{h}_2 \le \cdots \le \bar{h}_K$
- **Create difference r.v.s** $\bar{\Delta}_k = \bar{h}_1 - \bar{h}_k$
- **These** $\bar{\Delta}_k$ **have a joint t-distribution, approximately, and we could exploit that, but let's keep things simple, so …**

And this:

- **Reject** $\hat{\mathbf{x}}_k$ **as "bad" if confident that** $\bar{\Delta}_k < 0$
- **That is, put** $\hat{\mathbf{x}}_k \in \mathcal{X}^*$ **if not confident that**

$$Eh\left(\hat{\mathbf{x}}_k, \tilde{\mathbf{I}}\right) > Eh\left(\hat{\mathbf{x}}_1, \tilde{\mathbf{I}}\right) \quad \left(\geq Eh\left(\hat{\mathbf{x}}^*, \tilde{\mathbf{I}}\right)\right)$$

- **Let** $s_{\bar{\Delta}_k}$ **be the sample s.d. for estimate** $\bar{\Delta}_k$

- **"Accept"** $\hat{\mathbf{x}}_k$ **if the** $100\left(1-\frac{\alpha}{K-1}\right)\%$ **confidence interval on** $\bar{\Delta}_k$ **covers 0:** $\left(-\infty, \bar{\Delta}_k + z_{\frac{\alpha}{K-1}} s_{\bar{\Delta}_k}\right]$

So:

- Overall, we'll be 100$\times$(1– α)% confident that we have not rejected a good $\hat{\mathbf{x}}_k \in \mathcal{X}^*$
- Above procedure depends on Boole-Bonferroni inequality: not very strong.
- On the other hand, we used CRNs in comparing the $\hat{\mathbf{x}}_k$ so we have employed a useful variance-reduction technique. (1 or 2 orders of mag. improvement)
- Many variants/improvements possible

Testing step

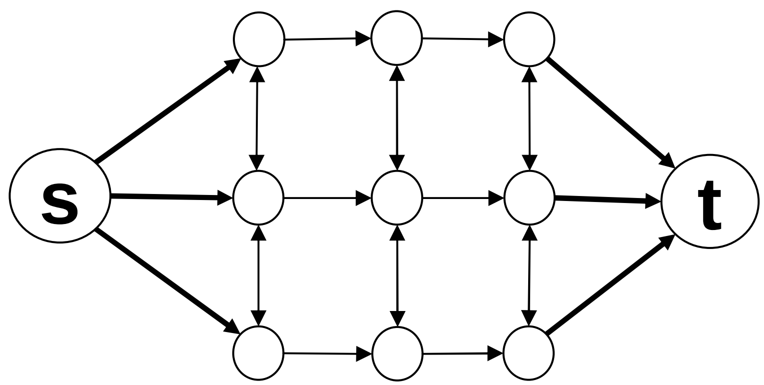
- Not an issue if $K^* = 1$.
- Will not cover in this talk, except to say that, empirically:
- All near-optimal solutions in this talk's test problems are within 2% of optimal with 95% confidence.

Advantages to BEST

- No large approximating problems with multiple scenarios to solve
- For the most part, we're solving simple bounding models and using Monte Carlo to evaluate 2SSPs with fixed first-stage variables
- No complicated decompositions needed

SMFI, computational results (1)

- **Grid network**



- **100 samples for each** $\hat{\mathbf{x}}_k$
- u_k **is uniform[10,100],** p_k**=0.9**
- **Only resource constraint:** $\sum_{k \in A} x_k \leq R$
- **Upper bound optimized**
- **VR for screening**
- **1 GHz laptop using GAMS/CPLEX**

SMFI, computation results (2)

- **95% confidence**

Grid	$\|A\|$	R	L	$\|\mathcal{X}\|$	$\|\mathcal{X}^*\|$
8×8	183	7	30	3	1
		8		7	1
		9		65	6
10×10	304	9		5	3
		10		9	2
20×12	715	12	100	19	1
20×20	1179	20		7	1

Stochastic plant location (SPL)

- **Uncertain demand for a single product**
- ***x_i* = 1 if plant *i* to be built, else 0**

$$\min_{\mathbf{x}\in X} Eh\left(\mathbf{x},\tilde{\mathbf{d}}\right) \text{ where } X = \left\{\mathbf{x}\in \{0,1\}^{|I|} \mid \sum_{i\in I} x_i = k\right\} \text{ and}$$

$$\text{where } h\left(\mathbf{x},\tilde{\mathbf{d}}\right) \equiv \mathbf{cx} + \min_{\mathbf{y}\geq \mathbf{0}}\{\mathbf{fy}+\mathbf{gw}\}$$

$$\text{s.t. } \sum_{j\in J} y_{ij} \leq u_i x_i \quad \forall \text{ plants } i\in I$$

$$\sum_{i\in I} y_{ij} + w_j \geq \tilde{d}_j \quad \forall \text{ customers } j\in J$$

$$y_{ij}\geq 0 \ \forall\ i,j; \quad w_j \geq 0 \ \forall\ j\in J$$

SPL, computational results (1)

- **10 candidate plants, choose 5**
- **20 customer zones (rvs)**
- **Demands uniform, ±v% of mean**
- **Probabilistic UB, Jensen LB =784.7**
- **v=10: ub=801.9, *K*=2, *K**=1, T=15.6**
- **v=20: ub=840.0, *K*=5, *K**=1, T=17.4**
- **v=40: ub=939.9, *K*=28, *K**=1, T=29.7**

SPL, computational results (2)

- 20 candidate plants, choose 10
- 50 customer zones (rvs)
- Demands uniform, ±v% of mean
- Probabilistic UB, Jensen LB =958.8
- v=10: ub= 966.2, K=3, K*=2, T=133
- v=20: ub=1006.3, K=72, K*=5, T=255
- v=40: ub=1155.1, K=51, K*=17, T=560[a]
- [a] LB improved to 1097.8

Extensions

BEST (PETS, PEST, BETS?) will work for any 2SSP provided that

- First-stage variables are binary or integers of modest magnitude,
- An optimistic bound is not too hard to compute, and
- For fixed **x**, Monte Carlo sampling is efficient.

- **For optimistic bounds, we use Jensen’s ineq. and restricted recourse**
- **Often, the global, pessimistic bound will be probabilistic**

Generalizations

- 2nd-stage integer variables OK
- Does not depend on distributions: If you can generate the rvs, BEST works
- So, dependent rvs OK
- Probabilistic LBs?
- Multi-stage???

BEST

To End

Other work

- Interdiction of communications networks: Physical attacks and cyber-attacks
- General models for system interdiction and defense
- General theoretical work on SPs
- Applications of SP: Sealift deployments subject to bio-attacks
- Integer programming