Stochastic Network Interdiction

Susan Sanchez
Kevin Wood kwood@nps.navy.mil
Operations Research Department
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Purpose of this talk

- Describe a deterministic max-flow network interdiction problem
- Describe stochastic variants
- Provide a new, simple solution methodology for the stochastic problems
- Discuss extensions to other interdiction problems and more general two-stage stochastic programs

Generic network interdiction problem

- **Using limited resources, attack an adversary's network so as to minimize the functionality of that network (to the adversary).**
- **Networks: Road, pipeline, comm**
- **Functionality: <u>Max</u> <u>flow</u>, shortest path, convoy movement, path existence**
- **Attacks: Aerial sorties, cruise missiles, special ops, interception**
- **Can generalize: "system interdiction"**

Max-flow interdiction

Basic Deterministic Model

on *G*=(*N*,*A*) with artificial arc *a* = (*t*,*s*)

$$\min_{\mathbf{x}\in X}\max_{\mathbf{y}} y_a$$

$$\text{s.t.} \sum_{k\in FS(i)} y_k - \sum_{k\in RS(i)} y_k = 0 \quad \forall i\in N$$

$$0 \le y_k \le u_k(1-x_k) \quad \forall k\in A-a$$

where $X = \left\{\mathbf{x}\in \{0,1\}^{|A|} \mid R\mathbf{x} \le \mathbf{r}\right\}$

Max-flow interdiction

Converts to

$$\min_{\mathbf{x}\in X}\ \min_{\boldsymbol{\pi},\boldsymbol{\theta}} \sum_{k\in A-a} u_k\theta_k$$

$$\text{s.t.}\quad \pi_i - \pi_j + x_k + \theta_k \geq \begin{cases} 1 & if\ \ k = a \\ 0 & \forall\, k = (i,j)\in A-a \end{cases}$$

$$\theta_k \geq 0 \ \ \forall\, k\in A-a$$

Interdiction under uncertainty

- **Uncertain <u>success</u> or data, SMFI:**

$$\min_{\mathbf{x}\in X} E\tilde{z}(\mathbf{x})$$

where $\tilde{z}(\mathbf{x}) \equiv \max_{\mathbf{y}\geq\mathbf{0}} y_a$

s.t. Flow balance, and

$$0 \leq y_k \leq u_k(1-\tilde{I}_k x_k)\forall k \in A-a$$

$$\text{where } \tilde{I}_k = \begin{cases} 1 \text{ if interdiction of } k \text{ is successful} \\ 0 \text{ otherwise} \end{cases}$$

Probability of kill

- Assume $p_k = E[\tilde{I}_k]$ is known
- Weaponeers know this stuff!
- Well…

Bound on z^*, pessimistic

- **New soln methodology needs bounds**
- **From Jensen's inequality, obtain a global upper bound given a "good" $\hat{\mathbf{x}}$:**

$$
\begin{aligned}
z^* &\le E\tilde{z}(\hat{\mathbf{x}}) \text{ for any } \hat{\mathbf{x}} \in X \\
&\le \bar{z}(\hat{\mathbf{x}}) \qquad\qquad (= \bar{z}^*) \\
&\equiv \max y_a \\
&\quad \text{s.t. Flow balance in } y \\
&\qquad 0 \le y_k \le u_k(1 - E[\tilde{I}_k]\hat{x}_k) \; \forall k \in A - a
\end{aligned}
$$

- **Can also use probabilistic bounds**

Bound on *z**, optimistic

- **Lower bound:**

$$z^* \equiv E\tilde{z}(\mathbf{x})$$

$$\leq \min_{\mathbf{x} \in X} \underline{z}(\mathbf{x})$$

$$= \min_{\mathbf{x} \in X, \boldsymbol{\pi}, \boldsymbol{\theta}} \sum_{k \in A-a} u_k \theta_k$$

$$\text{s.t.} \quad \pi_i - \pi_j + E[\tilde{I}_k] x_k + \theta_k \geq \begin{cases} 1 & \text{if } k = (t, s) \\ 0 & \forall k = (i, j) \in A - a \end{cases}$$

$$\theta_k \geq 0 \ \ \forall k \in A - a$$

Bounds on *z**: Comments

- Bounds can be improved by expanding in terms of conditional probabilities e.g., by conditioning on the number of successful interdictions.
- Can use probabilistic bounds.
- But, keep it simple for this talk.

Solution methodology, outline

- *Partial Enumeration*: Find all $\hat{\mathbf{x}}$ that might be optimal by using the bounds. This set of candidate solutions is $\hat{\mathbf{X}}$.
- *Then Screen*: Use Monte Carlo screening methods to identify the best, or the best few $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$.
- *PETS*

Fundamental theorem for PE

- **Theorem 1: $\hat{\mathbf{x}}$ can be optimal for SMFI only if $\underline{z}\left(\hat{\mathbf{x}}\right) \le \overline{z}*$.**
- **So can find a set of candidate solutions using the algorithm on the next slide.**
- **For simplicity, assume that the set of feasible interdiction plans defined by *X* has a cardinality constraint:**

$$\sum_{k \in A} x_k = R$$

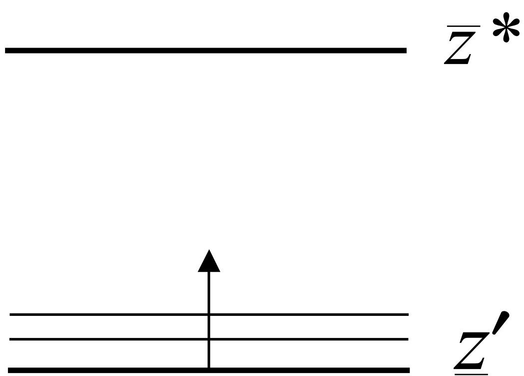
Alg. to find candidate solutions

1. $\hat{X}$ = $\varnothing$; Compute global UB $\bar{z}^*$;
2. Solve $\underline{z}' = \min_{\mathbf{x} \in X} \underline{z}(\mathbf{x})$ for $\hat{\mathbf{x}}$; /* $\underline{z}' = \underline{z}(\hat{\mathbf{x}})$ */
3. If $(\underline{z}' > \bar{z}^*)$ print $\hat{X}$ and halt;
4. Add $\hat{\mathbf{x}}$ to $\hat{X}$;
5. Add constraint $\sum_{k | \hat{x}_k = 1} x_k \le R - 1$

   to constraint set X and go to 1;

Alg. to find candidate solutions

Find good $\hat{\mathbf{x}}$



Compute $\bar{z}^* = \bar{z}\left(\hat{\mathbf{x}}\right)$

Compute $\underline{z}' = \min\limits_{\substack{\mathbf{x}\in X \\ \mathbf{x}\neq\hat{\mathbf{x}}_1 \\ \mathbf{x}\neq\hat{\mathbf{x}}_2 \\ \vdots}} \underline{z}\left(\mathbf{x}\right)$

Screening candidate solutions

- For small R, can actually compute Ez exactly because there are only 2^R ways that R attempted interdictions can be successful or fail
- But, will discuss and illustrate statistical screening procedures because they are necessary for typical applications of PETS

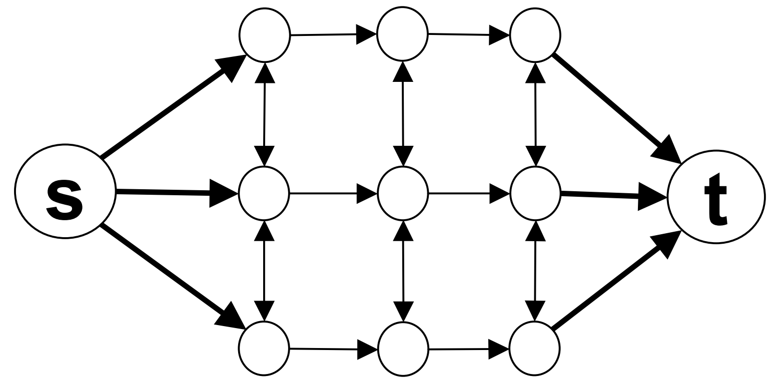
More on screening

Advantages to PETS

- No large approximating problems with multiple scenarios need be solved
- For the most part, we’re solving simple bounding models and using Monte Carlo to evaluate TSSPs with fixed first-stage variables
- No complicated decompositions needed

Computational results (1)

- **Grid networks**



- **100 samples for each** $\hat{\mathbf{x}}$
- $\boldsymbol{u_k}$ **is uniform[10,100],** $\boldsymbol{p_k}$**=0.9**
- **Only resource constraint:** $\sum_{k|} x_k \leq R$

- **500 MHz laptop using GAMS/OSL**

Table of results

10 by 10 grid, $|N|=102$, $|A|=304$

R	LB	Z_{best}	UB	Num. Soln.	Good Solns.

Computational results (2)

10 by 10 grid, |*N*|=102, |*A*|=310

95% confidence

R	LB	z_{best}	UB	Num. Solns.	Good Solns.
6	124.1	129.3	133.7	6	
7	98.6	103.3	108.8	20	
8	74.3	79.6	84.5	27	
9	55.3	61.9	66.8	41	

Extensions

- **PETS will work for any TSSP provided that**
  - First-stage variables are binary or integers of modest magnitude,
  - An optimistic bound is not too hard to compute, and
  - For fixed **x**, Monte Carlo sampling is efficient.
- **For optimistic bounds, we use Jensen's ineq. and restricted recourse**
- **Often, the global, pessimistic bound will be probabilistic**

Comments and Conclusions

- New, simple technique to solve stochastic network interdiction problems
- Generalizes to a broad class of TSSPs