<u>Darboux's formula</u> p polynomial $\deg p \le n \Rightarrow$

$$\boxed{\begin{aligned}&\sum_{k=0}^{n}(-1)^k(x-a)^k\left[p^{(n-k)}(1)f^{(k)}(x)-p^{(n-k)}(0)f^{(k)}(a)\right]=\\&=(-1)^n(x-a)^{n+1}\int_0^1 p(t)\,f^{(n+1)}(a+t(x-a))\,dt=:R_n.\end{aligned}}$$

Proof is to <u>integrate</u> <u>by</u> <u>parts</u> with $u=p(t)$, $dv=f^{(n+1)}(a+t(x-a))\,dt$. Get

$$R_n=(-1)^n(x-a)^n\left[p(1)f^{(n)}(x)-p(0)f^{(n)}(a)\right]+R_{n-1}$$

$$R_{n-1}=(-1)^{n-1}(x-a)^n\int_0^1 p'(t)f^{(n)}(a+t(x-a))\,dt.$$

Repeat until we get $R_{-1}=0$. In general

$$R_{n-k}=(-1)^{n-k}(x-a)^{n+1-k}\int_0^1 p^{(k)}(t)f^{(n+1-k)}(a+t(x-a))\,dt$$

and so

$$R_0=(x-a)\int_0^1 \underbrace{p^{(n)}(t)}f'(a+t(x-a))\,dt$$

$$u=\text{constant}\Rightarrow du=0=R_{-1}$$

$$v=\frac{f(a+t(x-a))}{x-a}$$

$$R_0=p^{(n)}(1)f(x)-p^{(n)}(0)f(a),$$

since $p^{(n)}(t)=$ constant.

<u>Integral</u> <u>form</u> <u>of</u> <u>remainder</u> <u>in</u> <u>Taylor</u> <u>series</u>.

$$p(t):=(t-1)^n$$

$$p'(t) = n(t-1)^{n-1}$$

$$p''(t) = n(n-1)(t-1)^{n-2}$$

$$\vdots$$

$$p^{(k)}(t) = n(n-1)\cdots(n-k+1)(t-1)^{n-k}$$

$$\vdots$$

$$p^{(n)}(t) \equiv n!$$

$$p^{(n)}(1) = n!$$

$$p^{(n-k)}(1) = 0, \quad 0 \le k \le n$$

$$p^{(k)}(1) \equiv 0, \quad 0 \le k < n$$

$$p^{(n)}(1) = n!$$

$$p^{(k)}(0) = (-1)^{n-k} n(n-1)\cdots(n-k+1)$$

$$= (-1)^{n-k} \frac{n!}{(n-k)!}$$

$$p^{(n-k)}(0) = (-1)^k \frac{n!}{k!}$$

Get

$$n!\, f(x) - \sum_{k=0}^{n} \cancel{(-1)^k}(x-a)^k \cancel{(-1)^k} \frac{n!}{k!} f^{(k)}(a) =$$

$$= (-1)^n (x-a)^{n+1} \int_0^1 (t-1)^n f^{(n+1)}(a+t(x-a))\, dt$$

that is

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_0^1 (1-t)^n f^{(n+1)}(a+t(x-a))\, dt\, (x-a)^{n+1}}$$

CoV: $s = a + t(x-a)$, $ds = (x-a)\,dt$

$$t = \frac{s-a}{x-a}, \quad 1-t = \frac{x-a-s+a}{x-a} = \frac{x-s}{x-a}$$

$$t=0 \Leftrightarrow s=a, \quad t=1 \Leftrightarrow s=x$$

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt}$$

The "weight function" $(x-t)^n$ is of constant sign on $[a, x]$. By the generalized mean value theorem for integrals

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{n!}\underbrace{\int_a^x (x-s)^n\,ds}_{(x-a)^{n+1}\int_0^1 (1-t)^n\,dt}$$

$$s = 1-t, \quad ds = -dt$$

$$= (x-a)^{n+1}\int_0^1 s^n\,ds$$

$$= \frac{(x-a)^{n+1}}{n+1}$$

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

c strictly between a and x.