Some convergent power series, easily obtained

⓪ **Lemma.** $\varepsilon_n = n^p a^n$, p real, $0 \le a < 1$
$\Rightarrow \varepsilon_n \to 0$ as $n \to +\infty$.

□ Trivial if $a = 0$; then $\varepsilon_n = 0$ for $n \geq 1$. Suppose $0 < a < 1$. Then

$$\ln \varepsilon_n = p \ln n + n \ln a$$

$$= n \underbrace{\ln a}_{<0} \left(1 + \frac{p}{\ln a} \underbrace{\frac{\ln n}{n}}_{\to 0} \right)$$

$$\to -\infty, \quad n \to +\infty$$

so

$$\varepsilon_n = e^{\ln \varepsilon_n} \to e^{-\infty} = 0. \quad \blacksquare$$

①

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}, \quad x \neq 1$$

same as

$$\frac{1}{1-x} - (1 + x + x^2 + \dots + x^n) = \frac{x^{n+1}}{1-x}, \quad x \neq 1$$

$$\to 0, \quad n \to +\infty$$

iff $|x| < 1$.

We write this as

$$\boxed{\begin{aligned}\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots, \quad |x|<1 \\ &= \sum_0^\infty x^n .\end{aligned}}$$

It is the <u>geometric series</u>. It converges to the function $1/(1-x)$ for $|x|<1$. In fact x can even be complex; we then call it z!

(2) Now differentiate the above equality and increase n by 1:

$$\frac{1}{(1-x)^2} - \left(1 + 2x + 3x^2 + \cdots + (n+1)x^n\right) =$$

$$= \frac{(1-x)(n+2)x^{n+1} + x^{n+2}}{(1-x)^2}$$

$$= \frac{(n+2)x^{n+1} - (n+1)x^{n+2}}{(1-x)^2}$$

$$= \frac{(n+2)x^{n+1}}{(1-x)^2}\left(1 - \underbrace{\frac{n+1}{n+2}}_{\to 1} x\right)$$

$$\sim \frac{(n+2)x^{n+1}}{1-x} \to 0, \quad n \to +\infty$$

iff $|x|<1$.

Thus, replacing x by $-x$,

$$\boxed{\begin{aligned}(1+x)^{-2} &= 1-2x+3x^2-4x^3+\cdots \\ &= \sum_0^\infty (-1)^n (n+1) x^n, \quad |x|<1\end{aligned}}.$$

We could differentiate again and get

$$\frac{d}{dx}(1+x)^{-2} = -2(1+x)^{-3} =$$

$$= -2+6x-12x^2+20x^3-\cdots,$$

$$|x|<1,$$

that is

$$\boxed{\begin{aligned}(1+x)^{-3} &= 1-3x+6x^2-10x^3+\cdots \\ &= \sum_0^\infty (-1)^n \frac{(n+1)(n+2)}{2} x^n, \\ &\qquad |x|<1.\end{aligned}}$$

We've omitted the tedious work with the remainder term but this one still $\to 0$ as $n \to +\infty$ iff $|x|<1$. These are special cases of the

binomial series, for $(1+x)^a$, which we'll do later.

(3) Now replace x by t in our basic geometric sum relation to get

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-1)^n t^n + \\ + (-1)^{n+1} \frac{t^{n+1}}{1+t}, \quad t \neq -1$$

Integrate from 0 to x (> -1) and make the simple CoV $s = 1+t$:

$$\int_0^x \frac{dt}{1+t} = \int_1^{1+x} \frac{dx}{s} = \ln(1+x) = \\ = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \\ + (-1)^n \int_0^x \frac{t^n}{1+t}\, dt, \quad 1+x > 0.$$

We have also decreased n by 1.

So we have

We want to show that

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \qquad -1 < x \leq 1}$$

Nota Bene
= works!

This will then imply that

$$\boxed{\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots}.$$

Thus we must show that

$$\int_0^x \frac{t^n}{1+t}\,dt \to 0, \quad n \to +\infty,$$

for $-1 < x \leq 1$.

The case when $x \geq 0$ is easy. Then $0 \leq t \leq x$ so $1 \leq 1+t$ and $\frac{1}{1+t} \leq 1$, $\frac{t^n}{1+t} \leq t^n$, so

$$0 \leq \int_0^x \frac{t^n}{1+t}\,dt \leq \int_0^x t^n\,dt = \frac{x^{n+1}}{n+1} \to 0, \quad n \to +\infty,$$

for $0 \leq x \leq 1$

If $x = 1$ this bound $\to 0$ very slowly. But also the series for $\ln 2$ converges very slowly.

Now let $-1 < x \le 0$. Then t satisfies $-1 < x \le t \le 0$ and so $0 < 1+x \le 1+t$ and $\frac{1}{1+t} \le \frac{1}{1+x}$.

Setting $s = -t$ we see that

$$\left| \int_0^x \frac{t^n}{1+t}\,dt \right| = \int_0^{-x} \frac{s^n}{1-s}\,ds$$

$$\le \frac{1}{1+x} \int_0^{-x} s^n\,ds$$

$$= \frac{1}{1+x} \frac{(-x)^{n+1}}{n+1}$$

$$= \frac{1}{1+x} \frac{|x|^{n+1}}{n+1}$$

$$\to 0, \quad n \to +\infty$$

if $-1 < x \le 0$.

In fact 'if' can be replaced by 'iff' since it is easy to

show, by the 'integral test', that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty .$$

We have just treated the logarithmic series.

(4) We now treat the series for atan x. We have

$$\frac{d}{dx} \operatorname{atan} x = \frac{1}{1+x^2}, \quad x \text{ real},$$

that is since $\operatorname{atan} 0 = 0$,

$$\operatorname{atan} x = \int_0^x \frac{dt}{1+t^2}, \quad x \text{ real}.$$

But our geometric sum formula gives ($t \leftarrow t^2$ and $n \leftarrow n-1$)

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^{n-1} t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}$$

Nota Bene $t^2 \geq 0$, not t!

Integration gives

$$\operatorname{atan} x = x - \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1}$$

$$+ (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt$$

All three functions of x here are odd, so $-x$ is the same as $+x$ except only for sign. We have

$$\left|\operatorname{atan} x - \left(x - \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right)\right| =$$

$$= \int_0^{|x|} \frac{t^{2n}}{1+t^2}\,dt \le \int_0^{|x|} t^{2n}\,dt =$$

$$\le \frac{|x|^{2n+1}}{2n+1} \to 0, \quad n \to +\infty$$

for $-1 \le x \le +1$

Nota Bene = in <u>both</u> places.

That is,

$$\boxed{\begin{aligned} \operatorname{atan} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \\ &= \sum_0^\infty (-1)^n \frac{x^{2n+1}}{2n+1}, \\ &\qquad -1 \le x \le +1 \end{aligned}}$$